Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

The Symbolic Notation of Aronhold and Clebsch.

BY WILLIAM F. OSGOOD, *Cambridge, Mass.*

In the 55th volume of Crelle, Aronhold communicated a symbolic notation for functional invariants of a general linear transformation, and applied this notation with marked success to the investigation of the ternary cubic. Clebsch then showed, in Crelle 58, p. 117 and ib. 59, and later in the Abh. d. k. Ges. d. Wiss. zu Göttingen, Bd. 17, that this notation suffices for the symbolic representation, in perspicuous form, of any functional invariant of simultaneous forms. The right to define functional invariants by means of certain symbolic expressions, well adapted to display their essential characteristics, was thus established.

This notation has been adopted generally among German mathematicians. As yet it has not, however, met with the acceptance in English literature to which its merits entitle it. The writer believes that this is due, in part, to the lack of an easily accessible and systematic exposition of the notation and of those fundamental properties of the same that render it so well adapted to the expression of functional invariants. The object of the present paper is to supply such an exposition.

§1.—*The Symbolic Notation for Binary Forms.*

Let the binary form of degree n be written

$$f_n(x) = \bar{a}_0 x_1^n + n_1 \bar{a}_1 x_1^{n-1} x_2 + n_2 \bar{a}_2 x_1^{n-2} x_2^2 + \dots .$$

where

$$n_k = \frac{n!}{(n-k)!\, k!}$$

This expression resembles the expansion of the binomial

$$(a_1 x_1 + a_2 x_2)^n = a_1^n x_1^n + n_1 a_1^{n-1} a_2 x_1^{n-1} x_2 + n_2 a_1^{n-2} a_2^2 x_1^{n-2} x_2^2 + \dots .$$

in its homogeneity in x, in the number of its terms and the degree of the latter in x_1 and x_2, and in its numerical coefficients. The $n+1$ coefficients $\bar{a}$ of $f_n(x)$

are, therefore, completely characterized by the $n+1$ products of powers of a_1 and a_2, and we may write *symbolically*

$$\bar{a}_0 = a_1^n, \qquad \bar{a}_1 = a_1^{n-1}a_2, \qquad \bar{a}_2 = a_1^{n-2}a_2^2, \dots . \qquad \bar{a}_k = a_1^{n-k}a_2^k, \dots . ;$$

$$f_n(x) = (a_1x_1 + a_2x_2)^n = a_x^n,$$

where a_x is an abbreviation for $a_1x_1 + a_2x_2$. Thus in any expression in which the $\bar{a}$'s enter linearly, they may be replaced by their symbolic representatives in the a's, for these latter symbolic products are linearly independent of each other. The symbols a_1, a_2, taken by themselves, have no meaning in terms of the coefficients $\bar{a}$; only when combined in expressions of degree n in a_1, a_2 are they capable of interpretation in terms of the $\bar{a}$'s.

A simple example of the application of this notation is the following. Let x be replaced by x' by means of the linear transformation

$$\left.\begin{aligned} x_1 &= \xi_1x_1' + \eta_1x_2' \\ x_2 &= \xi_2x_1' + \eta_2x_2' \end{aligned}\right\} \quad r = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix} \tag{1}$$

and let the form into which $f_n(x)$ is transformed be written

$$f_n'(x') = \bar{a}_0'x_1'^n + n_1\bar{a}_1'x_1'^{n-1}x_2' + n_2\bar{a}_2'x_1'^{n-2}x_2'^2 + \dots .$$

Then the coefficients $\bar{a}'$ are linear functions of the coefficients $\bar{a}$ and are homogeneous functions of degree n in the coefficients of the transformation ξ, η, for $f_n'(x') \equiv f_n(x)$. The relation between $\bar{a}'$ and $\bar{a}$ can be expressed symbolically in simple form. Let

$$f_n'(x') = (a_1'x_1' + a_2'x_2')^n = a_{x'}'^n,$$

so that $\qquad \bar{a}_0' = a_1'^n, \qquad \bar{a}_1' = a_1'^{n-1}a_2', \dots . \qquad \bar{a}_k' = a_1'^{n-k}a_2'^k.$

On the other hand,

$$a_1x_1 + a_2x_2 = a_1(\xi_1x_1' + \eta_1x_2') + a_2(\xi_2x_1' + \eta_2x_2') = a_\xi x_1' + a_\eta x_2',$$

and thus we have two symbolic expressions for $f_n'(x')$, namely $(a_1'x_1' + a_2'x_2')^n$ and $(a_\xi x_1' + a_\eta x_2')^n$. Hence we may put

$$a_1' = a_\xi, \quad a_2' = a_\eta, \tag{2}$$

and thus the coefficients $\bar{a}'$ can be represented symbolically in the simple form:

$$\bar{a}_0' = a_\xi^n, \qquad \bar{a}_1' = a_\xi^{n-1}a_\eta, \qquad \bar{a}_2' = a_\xi^{n-2}a_\eta, \dots . \qquad \bar{a}_k' = a_\xi^{n-k}a_\eta^k \dots .$$

The conciseness and clearness of expression that this example so well illustrates are characteristic features of this symbolic notation.

When it is desired to represent symbolically a function of the $\bar{a}$'s of higher degree than the first, confusion would arise from the direct introduction of the a's, since, for example, $a_1^n a_2^n = a_1^{n-i} a_2^i \,.\, a_1^i a_2^{n-i}$ would represent equally well any one of the products $\bar{a}_i \bar{a}_{n-i}$ $(i = 0,\ 1,\ 2,\ \ldots.)$. In order to avoid this ambiguity, further symbols b, c, ... are introduced, subject to the same conditions as the a's:

$$\bar{a}_k = a_1^{n-k} a_2^k = b_1^{n-k} b_2^k = c_1^{n-k} c_2^k = \ldots.\,,$$

$$f_n(x) = a_x^n = b_x^n = c_x^n = \ldots.$$

and, in a product of $\bar{a}$'s, each $\bar{a}$ is replaced by a different symbol. Thus $\bar{a}_1^2 \bar{a}_4 = a_1^{n-1} a_2 \,.\, b_1^{n-1} b_2 \,.\, c_1^{n-4} c_2^4 = a_1^{n-1} a_2 \,.\, c_1^{n-1} c_2 \,.\, b_1^{n-4} b_2^4 = \ldots.$ If we agree, therefore, to introduce symbols equal in number to the degree in the $\bar{a}$'s of the product we wish to represent, an ambiguity is no longer possible.

Suppose, for example, it be required to represent symbolically the discriminant Δ of the binary quadratic form

$$f_2(x) = \bar{a}_0 x_1^2 + 2\bar{a}_1 x_1 x_2 + \bar{a}_2 x_2^2 = a_x^2 = b_x^2 = c_x^2 = \ldots.$$

We have
$$\Delta = \begin{vmatrix} \bar{a}_0 & \bar{a}_1 \\ \bar{a}_1 & \bar{a}_2 \end{vmatrix} = \begin{vmatrix} a_1^2 & a_1 a_2 \\ b_1 b_2 & b_2^2 \end{vmatrix} = a_1 b_2 (ab),$$

where $(ab) = |a_1 b_2|$. But Δ might equally well have been expressed symbolically in the form:

$$\Delta = \begin{vmatrix} b_1^2 & b_1 b_2 \\ a_1 a_2 & a_2^2 \end{vmatrix} = b_1 a_2 (ba) = -\, b_1 a_2 (ab).$$

Adding these two expressions for Δ together, we have

$$2\Delta = (a_1 b_2 - a_2 b_1)(ab) = (ab)^2, \qquad \Delta = \tfrac{1}{2}(ab)^2.$$

From the symbolic expression thus obtained the invariant character of Δ with regard to linear transformation is at once evident. For, (1) and (2):

$$(a'b') = \begin{vmatrix} a_1\xi_1 + a_2\xi_2 & a_1\eta_1 + a_2\eta_2 \\ b_1\xi_1 + b_2\xi_2 & b_1\eta_1 + b_2\eta_2 \end{vmatrix} = r(ab),$$

and hence
$$(a'b')^2 = r^2 (ab)^2$$
$$\Delta' = r^2 \Delta.$$

Here, again, the simplicity and perspicuity of this notation are striking features. One sees at once, when Δ is written in the symbolic form, 1) that

Δ is an invariant, 2) that the power of r that enters is the second, 3) that the degree of Δ in the coefficients of f is the second (since Δ contains two symbols).*

§2.—*The Symbolic Expression of Invariants of Binary Forms.*

From the purely formal relations between the symbols

$$\left.\begin{array}{r} a'_{x'} = a_x \\ (a'b') = r(ab) \end{array}\right\} \qquad (3)$$

it is at once evident that invariants can be written down at pleasure. Thus

$$(ab)^{2i}a_x^{n-2i}b_x^{n-2i}, \qquad (ab)^2(bc)^2(ac)^2a_x^{n-4}b_x^{n-4}c_x^{n-4}, \qquad (ab)^n$$

are certainly invariants (I include here under this expression covariants as well). The last one vanishes identically when n is odd, for then, by an interchange of the equivalent symbols a and b,

$$(ab)^n = (ba)^n = -(ab)^n, \quad 2(ab)^n = 0,$$

and it is shown in like manner that any invariant that changes sign when equivalent symbols are interchanged vanishes identically, for example

$$(ab)^{2i+1}a_x^{n-2i-1}b_x^{n-2i-1} \equiv 0.$$

To obtain an invariant, therefore, it is only necessary to write down a product of factors of the above types (3)

$$\Pi\{(ab),\ (bc),\ \ldots.\ a_x,\ b_x,\ \ldots.\}$$

subject to the condition that each symbol $a, b, \ldots.$ shall appear just n times. The number, λ, of the determinant factors in Π will be the exponent of r in the invariant equation

$$\Pi' = \Pi\{(a'b'),\ (b'c'),\ \ldots.\ a'_x,\ b'_x,\ \ldots.\} = r^\lambda\Pi\{(ab),\ (bc)\ \ldots.a_x,\ b_x\ \ldots.\}$$

Furthermore, if $\Pi_1, \Pi_2, \Pi_3, \ldots.$ are such products, all having the same number of symbols of f, the same degree in x, and the same λ, then

$$P = \varkappa_1\Pi_1 + \varkappa_2\Pi_2 + \varkappa_3\Pi_3 + \ldots.$$

where $\varkappa_1, \varkappa_2, \varkappa_3, \ldots.$ are arbitrary numerical constants, will also be an invariant, $P' = r^\lambda P$.

---

* For a further treatment of the subject of this paragraph see Gordan's *Vorlesungen über Invariantentheorie*, edited by Kerschensteiner, Leipzig, 1885, Vol. II, where this symbolic notation is used throughout; Clebsch's *Theorie der algebraischen binären Formen*, Leipzig, 1872; Clebsch's *Vorlesungen über Geometrie*, edited by Lindemann, Leipzig, 1876, Vol. I, p. 167.

Thus far we have considered only a single form $f_n(x)$. The extension to the case of simultaneous forms $f_n(x)$, $\phi_m(x)$, ... presents no difficulty. Let $\phi_m(x) = \alpha_x^m = \beta_x^m = \dots$. Then in the symbolic product

$$\Pi\{(ab)\dots, (a\alpha)\dots, (\alpha\beta)\dots, a_x\dots, \alpha_x\dots\}$$

the symbols $a, b, \dots$ must each appear n times, the symbols $\alpha, \beta, \dots$ m times, etc. Π will then be a simultaneous invariant of f, ϕ, ..., and

$$P = \varkappa_1\Pi_1 + \varkappa_2\Pi_2 + \varkappa_3\Pi_3 + \dots,$$

formed subject to conditions similar to the previous, will also be a simultaneous invariant of f, ϕ, ...

We have thus seen how to form at pleasure, with the aid of this symbolic notation, invariants of a form $f_n(x)$ or of simultaneous forms $f_n(x)$, $\phi_m(x)$, ..., and the form of the expressions thus obtained was such that the essential properties of the functions of the coefficients of f, ϕ, ... that they represented were evident at the first glance. Now the fundamental property of this notation,—the property that renders it an adequate and well adapted means of investigation for the theory of invariants,—is the converse of the above, namely, that every invariant can be expressed in the form

$$P = \varkappa_1\Pi_1 + \varkappa_2\Pi_2 + \varkappa_3\Pi_3 + \dots$$

Stated as a theorem*: *Every rational integral invariant* (including covariants) *of the binary forms* $f_n(x) = a_x^n = b_x^n = \dots$, $\phi_m(x) = \alpha_x^m = \beta_x^m = \dots$, *etc., can be expressed as a rational integral function of factors* $(ab)\dots$, $(a\alpha)\dots$, $(\alpha\beta)\dots$, $a_x\dots$, $\alpha_x\dots$

§3.—*The Symbolic Notation for Ternary Forms.*

The extension of the foregoing symbolic notation to ternary forms is at once evident. Let

$$f_n(x) = \bar{a}_{n,0,0}x_1^n + n\bar{a}_{n-1,1,0}x_1^{n-1}x_2 + n(n-1)\bar{a}_{n-2,1,1}x_1^{n-2}x_2x_3 + \dots$$
$$+ \frac{n!}{i!\,j!\,k!}\bar{a}_{ijk}x_1^ix_2^jx_3^k\dots,$$

where $i + j + k = n$, be a ternary form of the n^{th} degree. Comparing it with the expansion of the trinomial

---

* An elementary proof of this theorem is to be found in Clebsch-Lindemann's *Geometrie*, p. 184. Other proofs are given in Clebsch's and Gordan's books previously referred to.

$$a_x^n = (a_1x_1 + a_2x_2 + a_3x_3)^n = a_1^nx_1^n + na_1^{n-1}a_2x_1^{n-1}x_2 + n(n-1)a_1^{n-2}a_2a_3x_1^{n-2}x_1x_2 + \dots\dots$$
$$+ \frac{n!}{i!\ j!\ k!}a_1^ia_2^ja_3^kx_1^ix_2^jx_3^k + \dots\dots,$$

where $i + j + k = n$ and a_x is written as an abbreviation for $a_1x_1 + a_2x_2 + a_3x_3$, we see that the coefficients $\bar{a}$ are completely characterized by the products of the a's, and we may, therefore, write symbolically

$$\bar{a}_{n,0,0} = a_1^n, \qquad \bar{a}_{n-1,1,0} = a_1^{n-1}a_2, \dots\dots \qquad \bar{a}_{ijk} = a_1^ia_2^ja_3^k \dots\dots$$

It is shown, similarly as before, that if, when a linear transformation of the x's is made:

$$x_i = \xi_ix_1' + \eta_ix_2' + \zeta_ix_3', \quad (i = 1,\ 2,\ 3), \quad r = |\xi_1\eta_2\zeta_3|,$$

the form, into which $f_n(x)$ goes is written

$$f_n'(x') = \sum\frac{n!}{i!\ j!\ k!}\bar{a}_{ijk}'x_1'^ix_2'^jx_3'^k = (a_1'x_1' + a_2'x_2' + a_3'x_3')^n = a_{x'}'^n,$$

then
$$a_1' = a_\xi, \quad a_2' = a_\eta, \quad a_3' = a_\zeta$$
and
$$\bar{a}_{ijk}' = a_\xi^ia_\eta^ja_\zeta^k.$$

When the function of the $\bar{a}$'s that is to be expressed symbolically is of higher degree than the first, additional symbols must be introduced. Thus if Δ is the discriminant of the quadratic

$$f_2(x) = \bar{a}_{200}x_1^2 + \bar{a}_{020}x_2^2 + \bar{a}_{002}x_3^2 + 2\bar{a}_{011}x_2x_3 + 2\bar{a}_{101}x_3x_1 + 2\bar{a}_{110}x_1x_2 = a_x^2 = b_x^2 = c_x^2 \dots\dots$$

we have
$$\Delta = \begin{vmatrix} \bar{a}_{200} & \bar{a}_{110} & \bar{a}_{101} \\ \bar{a}_{110} & \bar{a}_{020} & \bar{a}_{011} \\ \bar{a}_{101} & \bar{a}_{011} & \bar{a}_{002} \end{vmatrix} = \begin{vmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{vmatrix} = a_1b_2c_3(abc),$$

where (abc) denotes the determinant $|a_1b_2c_3|$. By interchanging the symbols a, b, c, and adding together the 6 expressions for Δ thus arising, we have

$$6\Delta = (abc)[a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3] = (abc)^2,$$
$$\Delta = \tfrac{1}{6}(abc)^2.$$

Let ρ, σ, τ be symbols of any form or forms,

$$f = a_x^n = b_x^n \dots\dots, \quad \phi = \alpha_x^m = \beta_x^m = \dots\dots, \text{ etc.},$$

ρ', σ', τ' symbols of the corresponding transformed forms f', ϕ', etc., so that $\rho_1' = \rho_\xi$, $\rho_2' = \rho_\eta$, $\rho_3' = \rho_\zeta$, etc. Then the law for the multiplication of determinants gives the relation

$$(\rho'\sigma'\tau') = r\,(\rho\sigma\tau).$$

From this it follows that

$$(a'b'c')^2 = r^2(abc)^2, \qquad \Delta' = r^2\Delta.$$

Thus far the analogy between the binary and the ternary forms has been a perfect one. But whereas, in the case of the former, it was necessary to consider only functions of a single variable, x, it is now necessary to consider, in addition, functions of u, where the u's may be interpreted geometrically as the coordinates of a line, the x's being taken as the coordinates of a point. The corresponding transformations of x and u are, since $u'_{x'} = u_x$,

$$\left.\begin{aligned} x_1 &= \xi_1 x_1' + \eta_1 x_2' + \zeta_1 x_3' \\ x_2 &= \xi_2 x_1' + \eta_2 x_2' + \zeta_2 x_3' \\ x_3 &= \xi_3 x_1' + \eta_3 x_2' + \zeta_3 x_3' \end{aligned}\right\}, \quad \left.\begin{aligned} u_1' &= \xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3 = u_\xi \\ u_2' &= \eta_1 u_1 + \eta_2 u_2 + \eta_3 u_3 = u_\eta \\ u_3' &= \zeta_1 u_1 + \zeta_2 u_2 + \zeta_3 u_3 = u_\zeta \end{aligned}\right\}, \quad r = |\xi_1 \eta_2 \zeta_3|$$

Hence it appears that, when x is linearly transformed, the line coordinates u and the symbols a, b, c are subjected to the same transformation. From this it follows that, whatever symbols ρ, σ may be,

$$(\rho'\sigma'u') = r(\rho\sigma u).$$

§4.—*The Symbolic Expression of Invariants of Ternary Forms.*

From the purely formal relations that have been obtained, namely,

$$\left.\begin{aligned} \rho'_{x'} &= \rho_x \\ (\rho'\sigma'\tau') &= r(\rho\sigma\tau) \\ (\rho'\sigma'u') &= r(\rho\sigma u) \end{aligned}\right\} \tag{4}$$

and with the help of the identical invariant u_x, invariants of the form f can be formed at pleasure. Thus for the cubic (cf. Clebsch-Lindemann, *Geometrie*, pp. 543 and 553),

$$f_3(x) = a_x^3 = b_x^3 = c_x^3 = \ldots,$$
$$S = (abc)(abd)(acd)(bcd),$$

is the invariant, whose vanishing is the condition that the Hessian of f,

$$\Delta = (abc)^2 a_x b_x c_x = 0,$$

break up into three straight lines, while

$$\Theta = (abu)^2 a_x b_x = 0$$

is the equation in line coordinates of the first polar of x, when x is taken as constant, or the locus of the points x, whose first polars touch a given line u, when u is taken as constant. The equation of f itself in line coordinates is

$$(abu)^2(cdu)^2(acu)(bdu) = 0\,.$$

In general, if

$$f = a_x^n = b_x^n = \dots\dots, \qquad \phi = \alpha_x^m = \beta_x^m = \dots\dots, \qquad \psi = \text{etc.},$$

be any simultaneous forms, an invariant of these forms can be obtained by multiplying together factors of the type (4) and the identical invariant u_x:

$$\Pi\{(abc)\dots\dots(a\alpha\beta)\dots\dots(\alpha\beta\gamma)\dots\dots(abu)\dots\dots(a\alpha u)\dots\dots(\alpha\beta u)\dots\dots a_x\dots\dots \alpha_x\dots\dots u_x\}\,,$$

care being taken that the symbols $a, b, \dots\dots$ appear just n times, the symbols $\alpha, \beta, \dots\dots$ just m times, etc. The number, λ, of determinant factors will be the exponent of r in the invariant equation

$$\Pi' = r^\lambda \Pi\,.$$

When there is only one form f, Π reduces to

$$\Pi\{(abc)\dots\dots(abu)\dots\dots a_x\dots\dots u_x\}\,.$$

If $\Pi_1, \Pi_2, \Pi_3 \dots\dots$ are such products, all having the same number of symbols of f, of ϕ, etc., the same degree in u and in x, and the same λ, then

$$P = \varkappa_1\Pi_1 + \varkappa_2\Pi_2 + \varkappa_3\Pi_3 \dots\dots,$$

where the $\varkappa$'s are arbitrary numerical constants, is also a simultaneous invariant of the forms f, $\phi \dots\dots$

Thus for the ternary as well as for the binary forms we have a symbolic notation, with the aid of which we can obtain invariants at pleasure, and the expression of the same is one that renders their chief characteristics, namely, their degree in the coefficients of each of the ground forms (shown by the number of symbols of each form that appear), in the point coordinates x, in the line coordinates u, and the value of λ (shown by the number of determinant factors), evident at the first glance. And just as in the case of the binary forms, so here, the converse is also true, namely, that every invariant of f can be expressed in the form*

$$P = \varkappa_1\Pi_1 + \varkappa_2\Pi_2 + \varkappa_3\Pi_3 + \dots\dots$$

---

* See the memoirs by Clebsch cited at the beginning of this paper; also Cl.-Lind., *Geom.*, p. 270. An elegant proof of this theorem is given in Gordan-Kerschensteiner, *Invariantentheorie*, II, §9, p. 110. The corresponding theorem for binary forms is first proved, and this proof is then extended to the case of n-ary forms.

Every rational integral invariant (including covariants, etc.) *of the ternary forms* $f = a_x^n = b_x^n = \ldots\ldots, \phi = \alpha_x^m = \beta_x^m \ldots\ldots, \psi =$ *etc., can be expressed as a rational integral function of the identical invariant,* u_x, *and of factors of the type* $(\rho\sigma\tau)$, $(\rho\sigma u)$, ρ_x, *where* ρ, σ, τ *are symbols of* f, ϕ, ...

§5.—*The Symbolic Representation of Higher Forms.*

The application of this symbolic notation to the representation of higher forms presents no difficulty. Thus it is evident that the biternary form can be written

$$(a_1x_1 + a_2x_2 + a_3x_3)^n(u_1\alpha_1 + u_2\alpha_2 + u_3\alpha_3)^m = a_x^n u_\alpha^m,$$

and the invariants of this form will be rational integral functions of factors of the type*

$$a_x, \quad (abu), \quad (abc), \quad a_\alpha, \quad u_\alpha, \quad (\alpha\beta x), \quad (\alpha\beta\gamma),$$

and of the identical invariant u_x. In the memoir in the Göttingen *Abhandlungen* above cited,† Clebsch proves that a ground form with several cogredient variables of each class (the number of the homogeneous point coordinates being n) can always be replaced by a system of ground forms (das reducirte äquivalente System), each containing at most but one variable from each class and such that the totality of the invariants of this system coincides with the totality of the invariants of the given ground form. Any simultaneous rational integral invariant of the forms of this system, and hence of the ground form itself, is expressible symbolically, rationally and integrally in terms of determinant factors of a few fundamental types, and thus Aronhold's symbolic notation is shown to have the same fundamental properties in the case of the higher forms as for the binary and ternary forms.‡

§6.—*On Certain Identical Relations.*

If, in the identically vanishing determinant,

$$\begin{vmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \end{vmatrix} \equiv 0,$$

---

* Cf. Cl.-Lind., *Geom.*, p. 942.

† Cf. also Cl.-Lind., *Geom.*, pp. 269 and 924.

‡ See also the reference to Gordan-Kerschensteiner given at the end of §4.

the first column, multiplied by x_1, and the second column, multiplied by x_2, be added to the third, the determinant thus arising can be expanded in the following form:

$$\begin{vmatrix} a_1 & a_2 & a_x \\ b_1 & b_2 & b_x \\ c_1 & c_2 & c_x \end{vmatrix} \equiv (ab)c_x + (bc)a_x + (ca)b_x \equiv 0\,. \qquad \text{I}$$

If, in this identity, we put $x_1 = d_2$, $x_2 = -d_1$, we have

$$(ab)(cd) + (bc)(ad) + (ca)(bd) \equiv 0\,. \qquad \text{II}$$

Again, if $c_1 = y_2$, $c_2 = -y_1$, I assumes the form

$$a_x b_y - a_y b_x = (ab)(xy)\,. \qquad \text{III}$$

Similar identities exist for the ternary forms. The development of

$$0 \equiv \begin{vmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & c_3 & 0 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_x \\ b_1 & b_2 & b_3 & b_x \\ c_1 & c_2 & c_3 & c_x \\ d_1 & d_2 & d_3 & d_x \end{vmatrix}$$

gives

$$(abc)d_x - (bcd)a_x + (cda)b_x - (dab)c_x \equiv 0\,, \qquad \text{IV}$$

and from this follows, by putting

$$x_1 = (e_2 f_3 - e_3 f_2) = (ef)_1, \quad x_2 = (e_3 f_1 - e_1 f_3) = (ef)_2, \quad x_3 = (e_1 f_2 - e_2 f_1) = (ef)_3,$$

$$(abc)(def) - (bcd)(aef) + (cda)(bef) - (dab)(cef) \equiv 0\,. \qquad \text{V}$$

The determinant

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

can be developed into two expressions, which, put equal to each other, give the identity

$$a_x b_y c_z + a_y b_z c_x + a_z b_x c_y - a_x b_z c_y - a_z b_y c_x - a_y b_x c_z = (abc)(xyz)\,. \qquad \text{VI}$$

Finally, the identities

$$a_x b_y - a_y b_x = (abu), \qquad \text{VII}$$

where

$$u_1 = (x_2 y_3 - x_3 y_2) = (xy)_1, \quad u_2 = (x_3 y_1 - x_1 y_3) = (xy)_2,$$
$$u_3 = (x_1 y_2 - x_2 y_1) = (xy)_3,$$

and $$a_x b_y - a_y b_x = (xyz), \qquad \text{VIII}$$

where $$z_1 = (ab)_1, \qquad z_2 = (ab)_2, \qquad z_3 = (ab)_3,$$

can be verified by actually multiplying out.

These identities are of prime importance in transforming symbolic expressions.*

CAMBRIDGE, MASS., *March*, 1892.

---

*The significance of these identities for binary forms is brought out in Gordan-Kerschensteiner, *Invariantentheorie*, II, Nos. 12 and 117, where it is shown that every relation between symbolic products (and hence, for example, between the invariants of a form or forms) can be referred directly to these identities. And the significance of these identities for ternary forms is no less fundamental.